MATH-SHU 362

Recitation 5: Series Solution of ODE

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Exercise 1. *Determine the radius of convergence of the given power series:*

1. $\sum_{n=0}^{\infty}(x-3)^n$;
2. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$;
3. $\sum_{n=0}^{\infty}\frac{(2x+1)^n}{n^2}$;
4. $\sum_{n=0}^{\infty}\left(\frac{2n}{e}\right)^{2n}\frac{x^{2n}}{(n!)^2}$.

Exercise 2. *Determine the Taylor series about the point x_0 for the given function:*

1. $f(x)=\sin x, x_0=0$;
2. $f(x)=\ln x, x_0=1$;
3. $f(x)=e^x, x_0=0$.

Exercise 3. *Prove that the bump function $f(x)$*

$$f(x)=\begin{cases} e^{-\frac{1}{1-x^2}} & |x|\leqslant 1, \\ 0 & |x|>1, \end{cases}$$

is smooth, of compact support, but is not analytic, i.e. cannot be written as Taylor's series.

Exercise 4. *Determine the coefficients a_n so that the equation*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0,$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

Exercise 5. *Prove that, for any absolute convergent series $(a_n)_{n\in\mathbb{N}}$, there exists another convergent series $(b_n)_{n\in\mathbb{N}}$ such that $\limsup_{n\to\infty}\frac{a_n}{b_n}=0$.*

Exercise 6. *Let $a_1=a_2=1$ and and $a_{n+2}=a_{n+1}+a_n, n\in\mathbb{N}$. This is the Fibonacci sequence. Let us find an explicit expression for a_n, $n\in\mathbb{N}$. Define $f(x)=\sum_{n=1}^{\infty}a_n x^n$.*

1. *Show that the power series has a positive radius of convergence.*
2. *Show that $f(x)=\frac{x}{1-x-x^2}$.*
3. *Write power series expansion for $f(x)$ and give the explicit value of a_n.*
4. *†† Prove that $gcd(a_n,a_m)=a_{gcd(n,m)}$. Here gcd means the greatest common divisor.*